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Nonlinear Aeroelasticity: The Steady-State Theory

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We present a rigorous analysis of the steady-state solution of a nonlinear aeroelastic system (as distinct from the flow alone) described by continuum models without resorting to numerical procedures. The wing structure model is that of a uniform beam (zero thickness) with 2 degrees of freedom: plunge and pitch. We consider typical section aerodynamics with inviscid and isentropic flow described by the nonlinear full potential equation of Euler for a nonzero angle of attack. The boundary conditions are the flow tangency condition and the Kutta–Joukowsky conditions. We show that nonzero time-invariant solutions exist at most for a discrete sequence of far-field flow speeds, which are determined by the torsion dynamics only. The reported U shapes of unmanned air vehicles can thus occur only for a discrete number of far-field speeds. They also exhibit a transonic dip, as a function of Mach number, for a nonzero angle of attack. The flow itself is shown to be made up of two parts, one in which shocks may be present but which generates no lift, and the other with no shocks but which is responsible for the lift. The first part cannot be linearized.

I. Introduction

THE central problem of aeroelasticity, or flutter, is a problem of instability (stability) of a structure in airflow, which requires, in particular, the specification of the equilibrium or steady state of the aeroelastic system. The rest, or zero, state for the structure with zero elastic energy and the steady flow of the air whatever the speed (which can be specified arbitrarily) is the generally accepted steady state, at least for a linear structure model, as here. In the absence of air flow, the structure will be at rest, whatever the shape. Here we model the wing structure as a uniform rectangular beam of zero thickness with 2 degrees of freedom: bending and twisting. At rest would mean that the displacement and the torsion angle are zero. The question is, are there other (that is, nonzero) steady-state solutions for aeroelastic equations? We show that such solutions cannot exist for arbitrary flow speeds, but only for a sequence of speeds depending on the structure torsion parameters and the Mach number. In the linearized case (linearized about the zero structure state), this is the sequence of "divergence" speeds.

It should be emphasized that the goal of this paper is to provide a rigorous analysis without resorting to purely numerical procedures, and needs perforce some idealization. For example, we assume that the structure is a uniform thin beam, omitting the effect of a varying airfoil thickness as of secondary importance.

Previous work, notably by Williams [1] following Landahl [2], has shown that flows (inviscid, as here) with shocks "cannot be linearized" but, even though no structure model is included, concludes that "the failure has little influence on the sectional characteristics of the wing." Here we show that the steady flow can be decomposed into two parts. One part, which may have shocks but which generates no lift, has only the trivial or zero linearization. The other part has no shocks but generates lift and has nonzero linearization at the divergence speed. The steady-state problem turns out to be a nonlinear eigenvalue problem for the aeroelastic torsion equation.

The steady-state (zero-frequency) equations are derived in Sec. II, following the unsteady version in [3]. The linearization (about the zero state) equation is solved in Sec. III. The nonlinear structure steady-state equation is solved in Sec. IV, where we show how it is reduced to an eigenvalue problem involving only the torsion angle dynamics. The zero-lift flow solution is developed in Sec. V. Conclusions are in Sec. VI.

II. Steady-State Aeroelastic System Equations

For a full statement of the aeroelastic system dynamic equations and boundary conditions, reference may be made to [3,4]. Here we shall only be concerned with the steady-state or time-invariant version, where all time derivatives therein are set to zero.

Thus, we have for the structure a uniform rectangular beam (see Fig. 1) with $h(\cdot)$ denoting the plunge and $\theta(\cdot)$ the pitch about an elastic axis as shown in the figure. We have, with $L(\cdot)$ denoting the steady-state aerodynamic lift and $M(\cdot)$ the moment:

$$EI\frac{\partial^4}{\partial y^4}h(y) = L(y), \qquad 0 < y < \ell \tag{1}$$

$$-GJ\frac{\partial^2}{\partial y^2}\theta(y) = M(y), \qquad 0 < y < \ell \tag{2}$$

$$\theta(0) = 0; \qquad h(0) = 0; \qquad h'(0) = 0; \qquad h'''(\ell) = 0; \qquad \theta'(\ell) = 0 \tag{3}$$

and there are no coupling terms anymore on the left-hand side of the equation.

The main problem is, of course, the calculation of the aerodynamic loading, the lift $L(y)$, and the moment $M(y)$. The aerodynamics is inviscid but isentropic, so that we use the Euler full potential equation [5] and specialize to typical section theory. Then $\phi(x,z)$, the velocity potential, satisfies

$$\begin{aligned}0 = {} & a_\infty^2\nabla^2\phi + \frac{(\gamma-1)}{2}|\nabla\phi_\infty - \nabla\phi|^2\nabla^2\phi \\ & - \left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2} - \left(\frac{\partial\phi}{\partial z}\right)^2\frac{\partial^2\phi}{\partial z^2} \\ & - 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial x\partial z}, \qquad -\infty < x, z < \infty, z \neq 0\end{aligned} \tag{4}$$

where $\phi_\infty(x,z) = U(x\cos\alpha + z\sin\alpha)$ is the far-field potential, α is the angle of attack, a_∞ is the speed of sound, and $M = \frac{U}{a_\infty} < 1$ with $\gamma > 1$ the ratio of specific heats, c_p, c_v.

The crucial structure–air interaction conditions are as follows:

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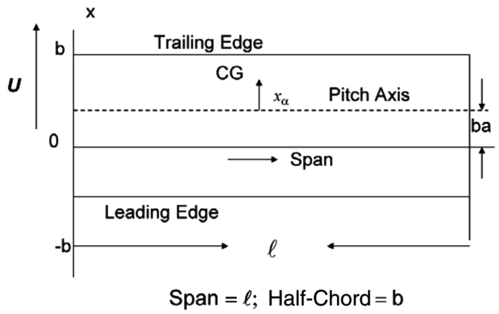

Fig. 1 Wing structure beam model.

1) Flow tangency along wing:

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = \frac{\partial\phi_\infty}{\partial z} + \nabla\phi\cdot\nabla z, \qquad |x| < b$$

where

$$z = -h(y) - (x-a)\theta(y), \qquad |x| < b, 0 < y < \ell$$

To allow for possible discontinuity in the flow above and below the wing (especially for a nonzero angle of attack), this condition must be split in two:

$$\frac{\partial\phi}{\partial z}(x, 0+) = U\sin\alpha + \frac{\partial\phi}{\partial x}(x, 0+)[-\theta(y)], \qquad |x| < b \tag{5}$$

$$\frac{\partial\phi}{\partial z}(x, 0-) = U\sin\alpha + \frac{\partial\phi}{\partial x}(x, 0-)[-\theta(y)], \qquad |x| < b \tag{6}$$

Note that at this level $\theta(y)$ is simply a given parameter.

2) Kutta–Joukowsky (zero pressure jump off the beam and at the trailing edge): Denoting the pressure by $p(x, z)$ and the steady-state acceleration potential by $\psi(x, z)$ (see [4] for definition), we have

$$\delta p(x) = p(x, 0+) - p(x, 0-) \qquad \delta p(x) = 0 \qquad |x| > b$$

$$\delta p(x) \to 0 \quad \text{as } x \to b-$$

Because, in the stationary case,

$$\delta p = -\rho_\infty\delta\psi \qquad \psi = \tfrac{1}{2}|\nabla\phi|^2$$

we may state the equivalent condition in terms of the acceleration potential:

$$\delta\psi(x) = 0, \qquad |x| > b \qquad \delta\psi(x) \to 0, \qquad |x| \to b-$$

For nonzero angle of attack $\alpha \neq 0$, we shall assume the seemingly stronger condition that the jump in the axial velocity and normal velocity both vanish.

$$\delta\left[\left(\frac{\partial\phi}{\partial x}\right)^2\right] = 0 = \delta\left[\left(\frac{\partial\phi}{\partial z}\right)^2\right], \qquad |x| < \infty \tag{7}$$

Finally,

$$L(y) = -\rho_\infty\int_{-b}^{b}\delta\psi\,\mathrm{d}x, \qquad 0 < \ell < y$$

$$M(y) = -\rho U\int_{-b}^{b}(x - ab)\delta\psi\,\mathrm{d}x, \qquad 0 < \ell < y$$

Now we formally define the "equilibrium" state.

Proposition 2.1:

$$h(y) = 0 = \theta(y), \qquad 0 < y < \ell, \qquad \phi(x, z) = \phi_\infty(x, z),$$
$$-\infty < x, z < \infty$$

is a solution to this set of equations for arbitrary U, α. We shall call this the equilibrium state.

Proof: By direct substitution.

III. Steady-State Solution: Linearized Equation

We wish now to examine the question of the existence of solutions to the steady-state aeroelastic equations other than the "equilibrium" solution of Theorem 2.1.

First we consider the linearized equations linearized about the equilibrium state. We only need to linearize the aerodynamic equations. We note that in the aerodynamic equation the structure state variables appear only in the flow tangency boundary conditions (5) and (6), and, in fact, involve only $\theta(y)$, which may be treated as a parameter. We assume that the solution $\phi(x, z)$ is analytic in $\theta(y)$, or, more conveniently, we consider $\lambda\theta(y)$ where we now allow λ as a complex variable and write the corresponding solution as $\phi(\lambda, x, z)$. We assume that $\phi(\lambda, x, z)$ is analytic in λ, that is, it has a power series expansion in λ in the finite complex plane about $\lambda = 0$.

Then

$$\phi(0, x, z) = \phi_\infty(x, z)$$

Define

$$\phi_1(x, z) = \frac{\partial}{\partial\lambda}\phi(\lambda, x, z)|_{\lambda=0}$$

Then $\phi_1(\cdot,\cdot)$ satisfies the familiar linearized field equation [3,4] extended to nonzero angle of attack α:

$$0 = a_\infty^2\left[\nabla^2\phi_1 - M^2\cos^2\alpha\frac{\partial^2\phi_1}{\partial x^2} - M^2\sin^2\alpha\frac{\partial^2\phi_1}{\partial z^2} - 2M^2\sin\alpha\cos\alpha\frac{\partial^2\phi_1}{\partial x\partial z}\right], \qquad -\infty < x, z < \infty, z \neq 0 \tag{8}$$

The main difference for nonzero α is the occurrence of the mixed partial derivative. This turns out, in fact, to be a significant difference.

The boundary conditions correspondingly at $z = 0$ are as follows:

1) Flow tangency: Equation (5) becomes

$$\frac{\partial\phi}{\partial z}(\lambda, x, 0+) = U\sin\alpha - \lambda\theta(y)\frac{\partial\phi}{\partial x}(\lambda, x, 0+), \qquad |x| < b \tag{9}$$

$$\frac{\partial\phi}{\partial z}(\lambda, x, 0-) = U\sin\alpha - \lambda\theta(y)\frac{\partial\phi}{\partial x}(\lambda, x, 0-), \qquad |x| < b \tag{10}$$

Hence,

$$\frac{\partial\phi_1}{\partial z}(x, 0+) = -\theta(y)U\cos\alpha = \frac{\partial\phi_1}{\partial z}(x, 0-), \qquad |x| < b \tag{11}$$

2) Kutta–Joukowsky: Equation (7) becomes

$$\psi_1 = \frac{\partial}{\partial\lambda}\psi(\lambda, x, z)|_{\lambda=0} = q_\infty\cdot\nabla\phi_1$$

Hence, we can calculate that

$$\delta\psi_1 = U\cos\alpha\left(\left.\frac{\partial\phi_1}{\partial x}\right|_{z=0+} - \left.\frac{\partial\phi_1}{\partial x}\right|_{z=0-}\right) + U\sin\alpha\left(\left.\frac{\partial\phi_1}{\partial z}\right|_{z=0+} - \left.\frac{\partial\phi_1}{\partial z}\right|_{z=0-}\right)$$
$$= U\cos\alpha\left[\frac{\partial\phi_1}{\partial x}(x, 0+) - \frac{\partial\phi_1}{\partial x}(x, 0-)\right]$$

for $|x| < b$, by Eq. (11), and

$$= 0, \quad \text{for } |x| > b$$

Next we proceed as in [3] and take spatial Fourier transforms $(L_p - L_q)$, $1 < p < 2$.

$$\tilde{\phi}_1(i\omega, z) = \int_{-\infty}^{\infty} e^{-i\omega x}\phi_1(x, z)\,\mathrm{d}x, \qquad -\infty < \omega < \infty$$

Then, as in [6], by taking Fourier transforms of Eq. (8), we have from the field equation

$$\tilde{\phi}_1(i\omega, z) = \frac{\tilde{\nu}(i\omega)}{r_1} e^{r_1 z}, \qquad z > 0 \tag{12}$$

$$\tilde{\phi}_1(i\omega, z) = \frac{\tilde{\nu}(i\omega)}{r_2} e^{r_2 z}, \qquad z < 0 \tag{13}$$

where now

$$\tilde{\nu}(i\omega) = \frac{\partial}{\partial z}\tilde{\phi}_1(i\omega, z)|_{z=0+} = \frac{\partial}{\partial z}\tilde{\phi}_1(i\omega, z)|_{z=0-}, \qquad -\infty < \omega < \infty$$

and

$$\frac{1}{r_1} = \frac{1 - M^2\sin^2\alpha}{M^2 i\omega \sin\alpha\cos\alpha - \sqrt{\omega^2(1 - M^2)}}$$

$$\frac{1}{r_2} = \frac{1 - M^2\sin^2\alpha}{M^2 i\omega \sin\alpha\cos\alpha + \sqrt{\omega^2(1 - M^2)}}$$

$$\bar{r}_1 = -r_2$$

Defining

$$A_1(x) = -\frac{\delta\psi_1}{U}, = -\cos\alpha\left[\frac{\partial\phi_1}{\partial x}(x, 0+) - \frac{\partial\phi_1}{\partial x}(x, 0-)\right],$$
$$|x| < b = 0, \qquad |x| > b$$

and

$$\hat{A}_1(i\omega) = \int_{-b}^{b} e^{-i\omega x}A_1(x)\,\mathrm{d}x, \qquad -\infty < \omega < \infty$$

we have the Possio equation (cf. [7]):

$$\hat{\nu}_1(i\omega) = \frac{1}{2}\frac{1}{\cos\alpha}\frac{1 - M^2\cos^2\alpha}{\sqrt{1 - M^2}}\frac{|\omega|}{i\omega}\hat{A}_1(i\omega), \qquad -\infty < \omega < \infty \tag{14}$$

which is the familiar airfoil equation and has the unique solution [7], using [8]:

$$A_1(x) = -2\theta U\cos\alpha\frac{\sqrt{1 - M^2}}{1 - M^2\cos^2\alpha}\sqrt{\frac{b - x}{b + x}}, \qquad |x| < b \tag{15}$$

Hence,

$$\tilde{\phi}_1(i\omega, z) = \frac{1}{2}\frac{1}{\cos\alpha}\frac{1 - M^2\cos^2\alpha}{\sqrt{1 - M^2}}\frac{|\omega|}{i\omega}\hat{A}_1(i\omega)\frac{1}{r_1}e^{+r_1 z}, \qquad z > 0 \tag{16}$$

$$= \frac{1}{2}\frac{1}{\cos\alpha}\frac{1 - M^2\cos^2\alpha}{\sqrt{1 - M^2}}\frac{|\omega|}{i\omega}\hat{A}_1(i\omega)\frac{1}{r_2}e^{+r_2 z}, \qquad z < 0 \tag{17}$$

For $\alpha = 0$, this yields

$$\frac{\partial\phi_1(x, z)}{\partial x} = \sqrt{1 - M^2}\int_{-b}^{b}\frac{z}{(1 - M^2)}\frac{A_1(\xi)\,\mathrm{d}\xi}{z^2 + (x - \xi)^2}, \qquad z \neq 0 \tag{18}$$

which we note is an odd function of the variable z.

The aerodynamic moment

$$M(y) = \rho U\int_{-b}^{b}(x - ab)A_1(x)\,\mathrm{d}x$$
$$= \theta(y)\pi\rho U^2 b^2(1 + 2a)\cos^2\alpha\frac{\sqrt{1 - M^2}}{1 - M^2\cos^2\alpha} \tag{19}$$

Letting

$$\mu^2 = \frac{\pi\rho b^2}{GJ}(1 + 2a)\frac{\sqrt{1 - M^2}}{1 - M^2\cos^2\alpha}\cos^2\alpha \tag{20}$$

we obtain

$$\theta''(y) = -\mu^2 U^2\theta(y), \qquad 0 < y < \ell \tag{21}$$

For free–free end conditions, we have a solution only for

$$\mu U = \frac{n\pi}{\ell}$$

(notice that the right side does not depend on M), where n is a nonnegative integer, with U_1 being the divergence speed [6]

$$U_1 = \sqrt{\pi}\frac{\sqrt{GJ}}{b\ell\sqrt{\rho(1 + 2a)\cos^2\alpha}}\cdot\left(\frac{1 - M^2\cos^2\alpha}{\sqrt{1 - M^2}}\right)^{1/2} \tag{22}$$

This yields, in particular, an analytical formula for the transonic "dip" in the divergence speed, as in [5], where the cantilever beam was considered. Note that the occurrence of the dip can be traced to the mixed partial derivative term in the field equation (8). With free–free end conditions, Eq. (21) yields the unique solution

$$\theta(y) = \theta(0)\cos\frac{n\pi}{\ell}y, \qquad 0 < y < \ell \tag{23}$$

The lift $L(\cdot)$, corresponding to $A_1(\cdot)$, is given by

$$\rho U\int_{-b}^{b}A_1(x)\,\mathrm{d}x = -2\pi\rho bU^2\cos\alpha\frac{\sqrt{1 - M^2}}{1 - M^2\cos^2\alpha}\theta(y) \tag{24}$$

and hence, the steady-state plunge equations become

$$EI\frac{\partial^4 h(y)}{\partial y^4} = -2\pi\rho bU^2\cos\alpha\frac{\sqrt{1 - M^2}}{1 - M^2\cos^2\alpha}\theta(y) \tag{25}$$

which by Eq. (22) does *not* depend on M, just like Eq. (23). Hence, the lift is completely determined by the steady-state pitch solution. Note that the pressure jump at the divergence speed does *not* depend on M.

IV. Nonlinear Structure Steady-State Solution

To determine the solution to the nonlinear equation, we continue the perturbation technique to obtain a series solution.

A. Series Solution

Continuing with the notation in Sec. III, let

$$\phi_k(x, z) = \frac{\partial^k}{\partial\lambda^k}\phi(\lambda, x, z)|_{\lambda=0} \tag{26}$$

so that

$$\phi(\lambda, x, z) = \sum_0 \frac{\lambda^k}{k!}\phi_k(x, z) \tag{27}$$

where, for $k \geq 2$, $\phi_k(\cdot, \cdot)$ satisfies the nonhomogeneous field equation

$$\nabla^2\phi_k - M^2\cos^2\alpha\frac{\partial^2\phi_k}{\partial x^2} - M^2\sin^2\alpha\frac{\partial^2\phi_k}{\partial z^2} - 2M^2\sin\alpha\cos\alpha\frac{\partial^2\phi_k}{\partial x\partial z}$$
$$= \frac{1}{a_\infty^2}g_k(x,z), \qquad -\infty < x, z < \infty \tag{28}$$

where $g_k(\cdot,\cdot)$ depends on $\phi_j(\cdot,\cdot)$, $j < k$. We shall return to this in more detail in Sec. V.

In particular, for $M = 0$, the field equation is linear, so that $g_k(\cdot,\cdot)$ is zero.

Next, the aerodynamic boundary conditions satisfied by $\phi_k(\cdot,\cdot)$ become, for $k \geq 2$:

1) Flow tangency condition Eqs. (5) and (6) yield for $|x| < b$:

$$\frac{\partial\phi_k}{\partial z}(x, 0+) = -k\theta(y)\frac{\partial\phi_{k-1}}{\partial x}(x, 0+) \tag{29}$$

$$\frac{\partial\phi_k}{\partial z}(x, 0-) = -k\theta(y)\frac{\partial\phi_{k-1}}{\partial x}(x, 0-) \tag{30}$$

which shows, in particular, that $\partial\phi_k/\partial z$ is *not* continuous at $z = 0$ for $k \geq 2$.

2) For the Kutta–Joukowsky condition, let

$$\psi_k(x,z) = \frac{\partial^k}{\partial\lambda^k}\psi(\lambda, x, z)|_{\lambda=0}$$

where

$$\psi(\lambda, x, z) = \frac{1}{2}|\nabla\phi(\lambda,\cdot)|^2$$

so that, by virtue of Eq. (7),

$$\delta\psi_k = U\cos\alpha\delta\left(\frac{\partial\phi_k}{\partial x}\right) + U\sin\alpha\delta\left(\frac{\partial\phi_k}{\partial z}\right) \tag{31}$$

Let

$$A_k(x) = -\frac{\delta\psi_k}{U}, \qquad |x| < b = 0, \qquad |x| > b$$

so that

$$\delta p_k(x) = -\rho U\delta\psi_k(x), \qquad = \rho U A_k(x), \qquad |x| < b \tag{32}$$

Hence, we require that

$$A_k(x) \to 0 \quad \text{as } x \to b-$$

and

$$A_k(x) = 0 \qquad |x| > b$$

By Eq. (31),

$$A_k = -\left[\cos\alpha\delta\left(\frac{\partial\phi_k}{\partial x}\right) + \sin\alpha\delta\left(\frac{\partial\phi_k}{\partial z}\right)\right], \qquad |x| < b \tag{33}$$

$$= 0, \qquad |x| > b \tag{34}$$

B. Decomposition of Solution

We show that we can construct the solution $\phi(\cdot,\cdot)$ for each $k \geq 2$ such that

$$\phi_k(x,z) = \phi_{L,k}(x,z) + \phi_{0,k}(x,z) \tag{35}$$

where we define

$$\psi_{L,k}(\cdot) = q_\infty\cdot\nabla\phi_{L,k} \qquad \psi_{0,k}(\cdot) = q_\infty\cdot\nabla\phi_{0,k}$$

so that

$$\psi_k(x,z) = \psi_{L,k}(x,z) + \psi_{0,k}(x,z)$$

For each k, $\phi_{L,k}(\cdot,\cdot)$ satisfies the homogeneous equation (28) (with $g_k(\cdot,\cdot)$ zero) and the boundary conditions

$$\frac{\partial\phi_{L,k}}{\partial z}(x, 0+) = -k\theta(y)\frac{\partial\phi_{L,k-1}}{\partial x}(x, 0+), \qquad |x| < b \tag{36}$$

$$\frac{\partial\phi_{L,k}}{\partial z}(x, 0-) = -k\theta(y)\frac{\partial\phi_{L,k-1}}{\partial x}(x, 0-), \qquad |x| < b \tag{37}$$

and

$$\delta\left(\frac{\partial\phi_{L,k}(x,\cdot)}{\partial x}\right) = 0 = \delta\left(\frac{\partial\phi_{L,k}(x,\cdot)}{\partial z}\right), \qquad |x| > b \tag{38}$$

which implies, in particular, that

$$\delta[\psi_{L,k}(x,\cdot)] = 0, \qquad |x| > b$$

Next, the flow potential $\phi_{0,k}(\cdot,\cdot)$ satisfies the nonhomogeneous Eq. (27) with the "zero" boundary conditions

$$\delta\left(\frac{\partial\phi_{0,k}(x,0)}{\partial z}\right) = 0, \qquad -\infty < x < \infty \tag{39}$$

$$\delta\left(\frac{\partial\phi_{0,k}(x,0)}{\partial x}\right) = 0, \qquad -\infty < x < \infty \tag{40}$$

so that, in particular,

$$\delta[\psi_{0,k}(x,\cdot)] = 0, \qquad |x| < b$$

and there is no aerodynamic lift or moment on the wing.

And finally,

$$\phi(\lambda, x, z) = \sum_0^\infty\frac{\lambda^k\phi_{L,k}(x,z)}{k!} + \sum_0^\infty\frac{\phi_{0,k}(x,z)\lambda^k}{k!} \tag{41}$$

C. Solving the Homogeneous Equation

Here we proceed along the lines in which we constructed the linear solution of $\phi_1(\cdot,\cdot)$. To reduce the writing effort, we will define

$$\varphi_k = \phi_{L,k} \qquad \psi_k = \psi_{L,k}$$

We use the notation

$$\nu_k(x,z) = \frac{\partial\varphi_k}{\partial z}(x,z) \qquad \gamma_k(x,z) = \frac{\partial\varphi_k}{\partial x}(x,z)$$

Then

$$\delta\psi_k(x) = U\cos\alpha\delta\gamma_k(x) + U\sin\alpha\delta\nu_k(x), \qquad |x| < b = 0,$$
$$|x| > b$$

where

$$\delta\psi_k(x) = \psi_k(x, 0+) - \psi_k(x, 0-) = 0, \qquad |x| > b$$

$$\delta\gamma_k(x) = \gamma_k(x, 0+) - \gamma_k(x, 0-) = 0, \qquad |x| > b$$

$$\delta\nu_k(x) = \nu_k(x, 0+) - \nu_k(x, 0-) = 0, \qquad |x| > b$$

Now, as in the linear case, we have from the homogeneous field equation, defining the $L_p - L_q$ Fourier transforms,

$$\tilde{\gamma}_k(i\omega, 0\pm) = \int_{-\infty}^{\infty}e^{-i\omega x}\tilde{\gamma}_k(x, 0\pm)\,\mathrm{d}x, \qquad -\infty < \omega < \infty$$

$$\tilde{\nu}_k(i\omega, 0\pm) = \int_{-\infty}^{\infty} e^{-i\omega x}\nu_k(x, 0\pm)\,\mathrm{d}x, \qquad -\infty < \omega < \infty$$

that

$$\tilde{\gamma}_k(i\omega, 0+) = \frac{i\omega(1 - M^2\sin^2\alpha)}{i\omega M^2\sin\alpha\cos\alpha - |\omega|\sqrt{1-M^2}}\tilde{\nu}_k(x, 0+)$$

$$\tilde{\gamma}_k(i\omega, 0-) = \frac{i\omega(1 - M^2\sin^2\alpha)}{i\omega M^2\sin\alpha\cos\alpha - |\omega|\sqrt{1-M^2}}\tilde{\nu}_k(x, 0-)$$

Or,

$$\left(M^2\sin\alpha\cos\alpha + \frac{|\omega|}{i\omega}\sqrt{1-M^2}\right)\tilde{\gamma}_k(i\omega, 0+) = (1 - M^2\sin^2\alpha\tilde{\nu}_k(i\omega, 0+) \tag{42}$$

$$\left(M^2\sin\alpha\cos\alpha - \frac{|\omega|}{i\omega}\sqrt{1-M^2}\right)\tilde{\gamma}_k(i\omega, 0-) = (1 - M^2\sin^2\alpha\tilde{\nu}_k(i\omega, 0-) \tag{43}$$

Hence, subtracting these equations, we have

$$\frac{|\omega|}{i\omega}[M^2\sin\alpha\cos\alpha\delta\tilde{\gamma}_k - (1 - M^2\sin^2\alpha)\delta\tilde{\nu}_k] = \sqrt{1-M^2}[\tilde{\gamma}_k(i\omega, 0+) + \tilde{\gamma}_k(i\omega, 0-)]$$

This is a Possio-type integral equation. With $\mathcal{T}$ denoting the Tricomi operator,

$$\mathcal{T}f = g; \qquad g(x) = \frac{1}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b}\sqrt{\frac{b+\xi}{b-\xi}}\frac{f(\xi)\,\mathrm{d}\xi}{\xi - x}, \qquad |x| < b$$

the solution as in [7] is given by

$$M^2\sin\alpha\cos\alpha\delta\gamma_k - (1 - M^2\sin^2\alpha)\delta\nu_k = \sqrt{1-M^2}\mathcal{T}(\bar{\gamma}_k) \tag{44}$$

where

$$\bar{\gamma}_k(x) = \gamma_k(x, 0+) + \gamma_k(x, 0-), \qquad |x| < b$$

Similarly, adding Eqs. (42) and (43), we have

$$\frac{|\omega|}{i\omega}\sqrt{1-M^2}\delta\tilde{\gamma}_k = (1 - M^2\sin^2\alpha)[\tilde{\nu}_k(i\omega, 0+) + \tilde{\nu}_k(i\omega, 0-)] - M^2\sin\alpha\cos\alpha[\tilde{\gamma}_k(i\omega, 0+) + \tilde{\gamma}_k(iw, 0-)]$$

and, analogous to Eq. (44),

$$\sqrt{1-M^2}\delta\gamma_k = (1 - M^2\sin^2\alpha)\mathcal{T}(\bar{\nu}_k) - M^2\sin\alpha\cos\alpha\mathcal{T}(\bar{\gamma}_k) \tag{45}$$

where

$$\bar{\nu}_k(x) = \nu_k(x, 0+) + \nu_k(x, 0-), \qquad |x| < b$$

Now the flow tangency conditions (36) and (37) yields

$$\bar{\nu}_k = -k\theta\bar{\gamma}_{k-1}, \qquad k \geq 2 \qquad \delta\nu_k = -k\theta\delta\gamma_{k-1}, \qquad k \geq 2$$

using which we have, from Eqs. (44) and (45), for $k \geq 2$,

$$M^2\sin\alpha\cos\alpha\delta\gamma_k + k\theta(1 - M^2\sin^2\alpha)\delta\gamma_{k-1} = \sqrt{1-M^2}\mathcal{T}(\bar{\gamma}_k) \tag{46}$$

$$\sqrt{1-M^2}\delta\gamma_k + k\theta(1 - M^2\sin^2\alpha)\mathcal{T}(\bar{\gamma}_{k-1}) + M^2\sin\alpha\cos\alpha\mathcal{T}(\bar{\gamma}_k) = 0 \tag{47}$$

Substituting Eq. (46) into Eq. (47), we obtain the difference equation valid for $k \geq 2$:

$$\delta\gamma_k(1 - M^2 + M^2\sin^2\alpha\cos^2\alpha) + \delta\gamma_{k-1}[+2k\theta M^2(1 - M^2\sin^2\alpha)\sin\alpha\cos\alpha] + \delta\gamma_{k-2}[+k(k-1)\theta^2(1 - M^2\sin^2\alpha)^2] = 0 \tag{48}$$

and by Eq. (33)

$$A_k = -\cos\alpha\delta\gamma_k + k\theta\sin\alpha\delta\gamma_{k-1}$$

Hence, defining

$$x_k = \left|\begin{matrix}\delta\gamma_k \\ -k\theta\delta\gamma_{k-1}\end{matrix}\right|, \qquad \delta\gamma_0 = 0$$

we have

$$x_k = k\theta A x_{k-1}, \qquad k \geq 2$$

where

$$A = \left|\begin{matrix}-2a_2 & a_3 \\ -1 & 0\end{matrix}\right|$$

where

$$a_2 = \frac{M^2}{1 + M^4\sin^2\alpha\cos^2\alpha - M^2}(1 - M^2\sin^2\alpha)\sin\alpha\cos\alpha \tag{49}$$

$$a_3 = \frac{1}{1 + M^4\sin^2\alpha\cos^2\alpha - M^2}(1 - M^2\sin^2\alpha)^2 \tag{50}$$

so that

$$\frac{A_k}{k!} = \theta^{k-1}|-\cos\alpha \quad \sin\alpha|A^{k-1}x_1, \qquad k \geq 2$$

and

$$\sum_2^\infty \frac{A_k}{k!} + A_1 = |-\cos\alpha \quad \sin\alpha|\left[\sum_2^\infty(\theta A)^{k-1}x_1 + x_1\right] = |-\cos\alpha \quad \sin\alpha|[(I - \theta A)^{-1}x_1] = \frac{1 + \theta\tan\alpha}{1 + 2a_2\theta + a_3\theta^2}A_1 \tag{51}$$

Note that now, in the form equation (51), the final answer does *not* require any smallness assumption on θ, so that this is no longer a "local solution."

Hence,

$$\phi_L(x, z) = \sum_0^\infty \frac{\phi_{L,k}(x, z)}{k!} = \phi_1(x, z) + \sum_2^\infty \frac{\phi_{L,k}(x, z)}{k!} = \frac{1 + \theta\tan\alpha}{1 + 2a_2\theta + a_3\theta^2}\phi_1(x, z) \tag{52}$$

where $\phi_1(x, z)$ satisfies the homogeneous equation (27) and contains no shocks in the flow off the wing, as is verified by the smoothness of the function $\phi_1(x, z)$ defined by Eqs. (16) and (17).

We can now calculate the corresponding aerodynamic moment

$$M(y) = \frac{1 + \theta(y)\tan\alpha}{1 + 2a_2\theta(y) + a_3\theta(y)^2}\int_{-b}^{b}\rho U(x - ab)A_1(x)\,\mathrm{d}x = \frac{1 + \theta(y)\tan\alpha}{1 + 2a_2\theta(y) + a_3\theta(y)^2}[-\mu^2U^2\theta(y)]$$

where μ^2 is given by Eq. (20). Correspondingly, we have the torsion equation

$$\begin{aligned}\theta''(y) &= -\mu^2U^2\theta(y)\cdot g[\theta(y)], \qquad 0 < y < \ell \\ \theta'(\ell) &= \theta'(0) = 0\end{aligned} \tag{53}$$

where

$$g(\theta) = \frac{1 + \theta \tan\alpha}{1 + 2a_2\theta + a_3\theta^2}$$

The lift $L(\cdot)$ is given by

$$L(y) = \frac{1 + \theta(y)\tan\alpha}{1 + 2a_2\theta(y) + a_3\theta(y)^2}\int_{-b}^{b} \rho U A_1(x)\,\mathrm{d}x$$
$$= g[\theta(y)](-2\pi\rho b U^2)\frac{\cos\alpha\sqrt{1-M^2}}{1 - M^2\cos^2\alpha}\theta(y)$$

by Eq. (23). Hence, the plunge equation is

$$EI\frac{\partial 4h(y)}{\partial y4} = -2\pi\rho b U^2\cos\alpha\frac{\sqrt{1-M^2}}{1-M^2\cos^2\alpha}\cdot\theta(y)\cdot g[\theta(y)],$$
$$0 < y < \ell \qquad h'(0) = 0 = h'(0) = h'''(\ell)$$

which is then completely determined by the solution to the torsion equation (53).

The main point now is that Eq. (53) is really a nonlinear eigenvalue problem, generalizing the linear case (cf. Sec. III). We have

$$\mu^2 U_n^2 = \gamma_n$$

where γ_n are the eigenvalues

$$\theta''(y) = -\gamma_n\theta(y)g[\theta(y)], \quad 0 < y < \ell \quad \theta'(0) = 0 = \theta'(\ell) \tag{54}$$

and we have solutions for, at most, a sequence $\{\gamma_n\}$ which depend on $\theta(0)$.

We need to show first that such a sequence $\{\gamma_n\}$ must be positive. We note that

$$-\int_0^\ell \theta''(y)\theta(y)\,\mathrm{d}y = \int_0^\ell \theta'(y)^2\,\mathrm{d}y = \gamma_n\int_0^\ell \theta(y)^2 g[\theta(y)]\,\mathrm{d}y \tag{55}$$

and

$$\int_0^\ell \theta''(y)\,\mathrm{d}y = 0 = \gamma_n\int_0^\ell \theta(y)g[\theta(y)]\,\mathrm{d}y \tag{56}$$

The denominator in $g(\theta)$ is positive for $M \leq 1$:

$$1 + 2a_2\theta(y) + a_3\theta(y)^2$$
$$= \frac{1 - M^2 + [M^2\sin\alpha\cos\alpha + \theta(y)(1 - M^2\sin^2\alpha)]^2}{(1 - M^2\sin^2\alpha)(1 - M^2\cos^2\alpha)} > 0 \tag{57}$$

If $\alpha = 0$, it is immediately clear that

$$g[\theta(y)] = \frac{1}{1 + (\theta^2/1 - M^2)}$$

and hence, from Eq. (55), it follows that

$$\gamma_n > 0$$

Also, dividing Eq. (54) by $\sqrt{1-M^2}$, we have

$$\frac{\theta''}{\sqrt{1-M^2}} = -\gamma_n\frac{\theta}{\sqrt{1-M^2}}\cdot\frac{1}{1 + (\theta^2/1 - M^2)}$$

and defining

$$\tilde{\theta} = \frac{\theta}{\sqrt{1-M^2}}$$

yields

$$\tilde{\theta}'' = -\gamma_n\tilde{\theta}\frac{1}{1 + \tilde{\theta}^2} \tag{58}$$

As we have noted, this is an eigenvalue problem and has a solution only for a discrete sequence $\gamma_n > 0$, which are determined as the zeros of an entire function of γ, which depends on $\theta(0)$.

For $\alpha \neq 0$, we can rewrite Eq. (53) as

$$\{1 - M^2 + [M^2\sin\alpha\cos\alpha + \theta(y)(1 - M^2\sin^2\alpha)]^2\}\theta''(y)$$
$$+ \mu_n\theta(y)[1 + \theta(y)\tan\alpha] = 0, \qquad 0 < y < \ell$$

where

$$\mu_n = \gamma_n(1 - M^2\sin^2\alpha)(1 - M^2\cos^2\alpha)$$

and the coefficient of $\theta''(\cdot)$ is nonzero for $0 < M < 1$. Once again, the μ_n are the zeros of an entire function, depending on $\theta(0)$. As $M \to 1$, Eq. (57) goes to the limit:

$$[1 + \theta(y)\cot\alpha]^2\sin^2\alpha\cos^2\alpha$$

for $\alpha \neq 0$ and, hence, we conclude that there is a transonic dip as $M \to 1$.

Finally, it may be of interest to note that Eqs. (55) and (56) are a special case of the nonlinear steady-state convolution–evolution equation in a Hilbert space, which characterizes nonlinear aeroelasticity (see [3]).

V. Solving the Zero-Lift Nonhomogeneous Flow Equation

We now turn to the solutions $\phi_{0,k}(\cdot,\cdot)$. For writing convenience, we will drop the zero and write $\phi_{0,k}$ as φ_k for this section.

We want to show that under suitable restrictions on the "forcing" functions $g_k(\cdot,\cdot)$, the nonhomogeneous Eq. (27) has a unique solution satisfying the zero boundary conditions (39) and (40), so that the flow generates *no* lift or moment on the wing and, hence, plays no role in the structure dynamics. The main feature of the solution will be that it exists in the weak or distributional sense, in particular, allowing for shocks in the flow. However, we shall not prove the existence of shocks in this paper.

The nonhomogeneous field equation is

$$\nabla^2\varphi_k - M^2\cos^2\alpha\frac{\partial^2\varphi_k}{\partial x^2} - M^2\sin^2\alpha\frac{\partial^2\varphi_k}{\partial z^2} - 2M^2\sin\alpha\cos\alpha\frac{\partial^2\varphi_k}{\partial x\partial z}$$
$$= \frac{1}{a_\infty^2}g_k(x,z), \qquad -\infty < x, z < \infty \tag{59}$$

and no discontinuities in the flow velocity at $z = 0$:

$$\delta\gamma_k = \frac{\partial\varphi_k}{\partial x}(x, 0+) - \frac{\partial\varphi_k}{\partial x}(x, 0-) = 0, \qquad -\infty < x < \infty \tag{60}$$

$$\delta\nu_k = \frac{\partial\varphi_k}{\partial z}(x, 0+) - \frac{\partial\varphi_k}{\partial z}(x, 0-) = 0, \qquad -\infty < x < \infty \tag{61}$$

The main thing here is to characterize the forcing function on the right of Eq. (59). We state this as a lemma.

Lemma 5.1:

$$g_k(x,z) = \sum\sum aij\sum_1^{k-1} C_\ell^k D^\ell\left(\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\right)\bigg|_{\lambda=0}\frac{\partial^2\phi_{k-\ell}}{\partial x_i\partial x_j}$$
$$+ \sum\sum bij\sum_2^{k-1} C_\ell^k D^\ell\left[\left(\frac{\partial\varphi}{\partial x_i}\right)^2\right]\bigg|_{\lambda=0}\frac{\partial^2\phi_{k-\ell}}{\partial x_j^2} \tag{62}$$

where C_ℓ^k denote binomial coefficients, and a_{ij}, b_{ij}, are given by writing Eq. (4) in the form

$$\sum_1^2\sum_1^2 aij\frac{\partial\phi}{\partial x_i}\cdot\frac{\partial\phi}{\partial x_j}\cdot\frac{\partial^2\phi}{\partial x_i\partial x_j} + \sum_1^2\sum_1^2 bij\left(\frac{\partial\varphi}{\partial x_i}\right)^2\left(\frac{\partial^2\phi}{\partial x_j^2}\right)$$
$$+ a_\infty^2\nabla^2\phi = 0 \tag{63}$$

where

$$x_1 = x \qquad x_2 = z; \qquad a_{ij} = 1; \qquad b_{ij} = \frac{\gamma - 1}{2}$$

Proof: With D denoting $\partial/\partial\lambda$, we have

$$D^k\left[\left(\frac{\partial\phi}{\partial x_i}\cdot\frac{\partial\phi}{\partial x_j}\right)\frac{\partial^2\phi}{\partial x_i x_j}\right]\bigg|_{\lambda=0}=\frac{\partial\phi_\infty}{\partial x_i}\cdot\frac{\partial\phi_\infty}{\partial x_j}\cdot\frac{\partial^2\phi_k}{\partial x_i\partial x_j} + \sum_{\ell=1}^{k-1}C_\ell^k D^\ell\left(\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\right)\bigg|_{\lambda=0}\frac{\partial^2\phi_{k-\ell}}{\partial x_i\partial x_j} \tag{64}$$

Hence, Eq. (62) follows.

As we have noted, $g_k(\cdot,\cdot)$ is defined iteratively and depends only on ϕ_j, $j<k$. For $k=2$, for example,

$$g_2(x,z)=2\sum_1^2\sum_1^2 aijq_i\frac{\partial\phi_1}{\partial x_1}\frac{\partial^2\phi_1}{\partial x_1\partial x_j} \tag{65}$$

$$q_1 = U\cos\alpha$$

$$q_2 = U\sin\alpha$$

where $\partial\phi(\cdot)/\partial x$ is given explicitly by Eq. (18) for a zero angle of attack and implicitly by Eqs. (16) and (17) for nonzero α. We note that $\phi_1(\cdot,\cdot)$ and its derivatives are bounded and vanish as $|x|\to\infty$, $|z|\to\infty$.

Hence, we can finally state

Theorem 5.1:

$$\varphi_k(x,z)=\frac{1}{2\pi}\frac{1}{1-M^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2[A^{-1}R,R]}\times\left[G_k(x-\xi,z-\eta)\frac{\partial}{\partial\xi}[A^{-1}R,R] + H_k(x-\xi,z-\eta)\frac{\partial}{\partial\eta}[A^{-1}R,R]\right]\mathrm{d}\xi\,\mathrm{d}\eta, \quad -\infty<x,z<\infty \tag{66}$$

where

$$G_k(x,z)=\int_{-\infty}^{x}g_k(\xi,z)\,\mathrm{d}\xi \qquad H_k(x,z)=\int_{-\infty}^{z}g_k(\xi,\eta)\,\mathrm{d}\eta$$

A is the 2×2 matrix

$$A=\begin{vmatrix}1-M^2\cos^2\alpha & M^2\sin\alpha\cos\alpha\\ M^2\sin\alpha\cos\alpha & 1-M^2\sin^2\alpha\end{vmatrix} \qquad \det A=(1-M^2)>0$$

$$R=\begin{vmatrix}\xi\\ \eta\end{vmatrix}$$

Proof: Let $\hat g(\cdot,\cdot)$ denote the Fourier transform

$$\hat g_k=(\omega_1,\omega_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g_k(x,z)e^{-i\omega_1 x}e^{-i\omega_2 z}\,\mathrm{d}x\,\mathrm{d}z$$

Let

$$\omega=\begin{vmatrix}\omega_1\\ \omega_2\end{vmatrix}$$

Then by taking the Fourier transform of both sides of Eq. (59) we have, formally,

$$[A\omega,\omega]\hat\varphi_k(\omega_1,\omega_2)=\hat g_k(\omega_1,\omega_2), \qquad -\infty<\omega_1,\omega_2<\infty \tag{67}$$

where $\hat\varphi(\cdot,\cdot)$ is the Fourier transform

$$\hat\varphi_k(\omega_1,\omega_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\omega_1 x}e^{-i\omega_2 z}\varphi_k(x,z)\,\mathrm{d}x\,\mathrm{d}z$$

and

$$[A\omega,\omega]>0 \quad \text{unless } \omega=0$$

Let c be a nonzero real number, then

$$c^2+[A\omega,\omega]\geq c^2>0$$

and (see [9])

$$\frac{1}{c^2+[A\omega,\omega]}$$

is the Fourier transform of the $L_1(R^2)$ function

$$K_0(c\sqrt{x^2+z^2}), \qquad -\infty<x,z<\infty$$

where $K_0(\cdot)$ is the Bessel K function of order 0.

Hence,

$$\hat\varphi_k(\omega_1,\omega_2)=\lim_{c\to 0}\frac{\hat g(\omega_1,\omega_2)}{c^2+[A\omega,\omega]}$$

and

$$\varphi_k(\omega_1,\omega_2)=\lim_{c\to 0}\frac{1}{2\pi(1-M^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}K_0(c\sqrt{[A^{-1}R,R]})\times g_k(x-\xi,z-\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{68}$$

Integrating by parts in the integral on the right,

$$\hat\varphi_k(\omega_1,\omega_2)=\lim_{c\to 0}\frac{1}{2\pi(1-M^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}c\frac{K_1(e\sqrt{[A^{-1}R,R]})}{2\sqrt{[A^{-1}R,R]}}\cdot\left[\frac{\partial}{\partial\xi}[A^{-1}R,R]G_k(x-\xi,z-\eta) + \frac{\partial}{\partial\eta}[A^{-1}R,R]H_k(x-\xi,z-\eta)\right]\mathrm{d}\xi\,\mathrm{d}\eta \tag{69}$$

Noting [9], in which the Bessel K function of order 1

$$K_1(x)=\frac{1}{x} \quad \text{as } x\to 0 \tag{70}$$

we obtain

$$[A^{-1}R,R]=\frac{1}{1-M^2}[(1-M^2\sin^2\alpha)x^2 + (1-M^2\cos^2\alpha)z^2-2M^2xz\cos\alpha\sin\alpha]$$

Hence, finally we have the series solution for the flow potential

$$\phi(x,z) \qquad -\infty<x,z<\infty=\phi_L(x,z)+\phi_0(1,x,z)$$

where

$$\phi_L(x,z)=\frac{1+\theta\tan\alpha}{1+2a_2\theta+a_3\theta^2}\phi_1(x,z) \tag{71}$$

$$\phi_0(\lambda,x,z)=\phi_\infty(x,z)+\sum_2^\infty\frac{\lambda^k\phi_{0,k}(x,z)}{k!} \tag{72}$$

We have thus a decomposition into two parts, one part, $\phi_L(\lambda,\cdot)$, which provides the lift and does not contain shocks, and the other part, $\phi_0(\lambda,\cdot)$, which does not provide lift but may contain shocks. Note that the linearization of $\phi_0(\lambda,\cdot)$ yields zero, consistent with the analysis of Williams [1] on flows with shocks, which follows the work of Landahl [2]. (See also [6] for a similar result for the unsteady transonic small disturbance equation.) Although we have an explicit expression for $\phi_L(x,z)$, so that the analycity assumption is not necessary, the precise radius of convergence of the series equation (72) is open (see also [5]).

Finally, it may be of interest to point out that the higher-order terms (higher than one) in the expansion may be considered as the static (or zero-frequency) analog to the expansion of a limit cycle oscillation in a series of harmonics of the fundamental frequency [5].

VI. Conclusions

Nonzero steady-state (zero-frequency/time-invariant) solutions of an aeroelastic system (slender high-aspect-ratio beam in an inviscid isentropic airflow) exist only at most for a sequence of far-field speeds, determined solely by the torsion dynamics and the angle of attack. The plunge motion is forced by the torsion. In particular, this would indicate that the reported steady-state U shape of unmanned air vehicles [10] can occur only at a discrete number of speeds. They also exhibit a transonic dip for a nonzero angle of attack.

The flow itself can be decomposed into two parts: one part that may have shocks but produces no lift and cannot be linearized (consistent with the conclusion of Williams in [1]), and another part that produces lift but has no shocks. The assumption of a zero thickness beam can be relaxed as in [11], which considers a thin plate of constant thickness, or even varying thickness, without changing the conclusion.

The nonlinear eigenvalue problem, which is shown to characterize the sequence of far-field speeds for which nonzero steady-state solutions exist, depends only on the structure (torsion) dynamics. Thus, the nonzero equilibrium, if any, for arbitrary far-field airspeed must arise solely from the nonlinearity of the structure, such as is considered for example in [8], still with a high-aspect-ratio beam model. In particular, we would expect this to be the case without restriction to typical section theory. Similar remarks apply to the hysteresis effects noted in [12], that these are not aerodynamic in origin, at least for inviscid flow.

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